The Structural Physical Approximation Conjecture

Fred Shultz[1]

Department of Mathematics, Wellesley College, Wellesley, Massachusetts 02481[a)]

It was conjectured that the structural physical approximation (SPA) of an optimal entanglement witness is separable (or equivalently, that the SPA of an optimal positive map is entanglement breaking). This conjecture was disproved, first for indecomposable maps and more recently for decomposable maps. The arguments in both cases are sketched along with important related results. This review includes background material on topics including entanglement witnesses, optimality, duality of cones, decomposability, and the statement and motivation for the SPA conjecture so that it should be accessible for a broad audience.

Resubmitted December 3, 2015

[a)]Electronic mail: fshultz@wellesley.edu

INTRODUCTION

Entanglement witnesses and positive maps are useful in detecting entanglement. For this purpose, positive maps are generally a more powerful tool than individual entanglement witnesses. For example, the transpose map detects entanglement of all entangled states in $M_2 \otimes M_2$ or $M_2 \otimes M_3$, while this is not the case for a single entanglement witness. However, entanglement witnesses are observables, hence can be implemented physically, while positive maps are not physically realizable unless they are completely positive. This led P. Horodecki[37], see also Ref. 38, to define the structural physical approximation (SPA) of a positive map to be a completely positive map formed by mixing the original map with as small an amount as possible of the completely depolarizing map. Mixing in the latter can be thought of as adding a minimal amount of a neutral disturbance, whose effects can be compensated for, since the completely depolarizing map takes every state to the maximally mixed state.

Lewenstein, Kraus, Cirac, and Horodecki[47] singled out those entanglement witnesses that are the most efficient in detecting entanglement, and called them optimal entanglement witnesses. Later Korbicz, Almeida, Bae, and Lewenstein[41] conjectured that the SPA of an optimal positive map would be entanglement breaking. Entanglement breaking maps have a particularly simple form which makes them straightforward to implement. Examples have been found by many investigators supporting this conjecture. Recently the conjecture was settled in the negative direction.

In this review we will begin by discussing background relevant to the SPA conjecture. We first review well known correspondences of linear maps from $\mathcal{A}_1$ to $\mathcal{A}_2$ with operators in $\mathcal{A}_1 \otimes \mathcal{A}_2$. We then discuss basics regarding entanglement witnesses, and the notion of decomposability of positive maps and entanglement witnesses. Finally we discuss optimality of entanglement witnesses, and the structural physical approximation of a positive map.

Then we state the structural physical approximation conjecture. We discuss the variety of examples found that support that conjecture. We then describe Ha and Kye's example[29] of an indecomposable entanglement witness that violates the SPA conjecture, and sketch their proof. Independently, in the same family of optimal entanglement witnesses studied by Ha and Kye, Størmer[71] by different methods proved that there is a witness that violates the SPA conjecture, which we also describe. Finally we discuss Chruściński and Sarbicki's example[18] of a decomposable entanglement witness that violates the conjecture.

We refer the reader interested in further background on entanglement witnesses and positive maps to the survey articles of Chruściński and Sarbicki[19], of Kye[44], and the book of Størmer[70].

Notation

We begin by fixing some notation and reviewing basic terminology. Let H_A and H_B denote finite dimensional Hilbert spaces, let $\mathcal{A}_1 = L(H_A)$ denote the linear operators on H_A, $\mathcal{A}_2 = L(H_B)$, and let $L(\mathcal{A}_1, \mathcal{A}_2)$ be the set of linear maps from $\mathcal{A}_1$ to $\mathcal{A}_2$. We identify $\mathcal{A}_1 \otimes \mathcal{A}_2$ with $L(H_A \otimes H_B)$. We will often identify H_A with $\mathbb{C}^m$ and H_B with $\mathbb{C}^n$, and denote the standard basis of $\mathbb{C}^m$ by $e_1, \ldots, e_m$. When convenient, we will identify $\mathcal{A}_1$ with M_m and $\mathcal{A}_2$ with M_n. We view $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_1 \otimes \mathcal{A}_2$ as Hilbert spaces with the Hilbert-Schmidt inner product $\langle X, Y \rangle = \operatorname{tr}(Y^\dagger X)$, where $\dagger$ denotes the Hermitian adjoint (or complex conjugate transpose as a matrix). For example, on $\mathcal{A}_1$ the Hermitian adjoint is given by

$$\langle Wx, y \rangle = \langle x, W^\dagger y \rangle \text{ for all } x, y \in H_A.$$

Similarly, if $\Phi \in L(\mathcal{A}_1, \mathcal{A}_2)$ then the dual map $\Phi^* : \mathcal{A}_2 \to \mathcal{A}_1$ is the linear map satisfying

$$\langle X, \Phi^*(Y) \rangle = \langle \Phi(X), Y \rangle \text{ for all } X \in \mathcal{A}_1, Y \in \mathcal{A}_2,$$

The transpose maps on $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_1 \otimes \mathcal{A}_2$ will be denoted by t. We denote the partial transpose map $I \otimes t$ by Γ. We note that $t^* = t$ and $\Gamma^* = \Gamma$.

A state on H is a positive (semi-definite) operator ρ in $L(H)$ with $\operatorname{tr} \rho = 1$. An operator A on $H_A \otimes H_B$ is *separable* if it can be expressed as a finite sum $A = \sum_i B_i \otimes C_i$ with $B_i \geq 0$ and $C_i \geq 0$. It follows that if ρ is a state on $H_A \otimes H_B$, then ρ is separable iff it is a convex combination of product states: $\rho = \sum_i t_i \sigma_i \otimes \tau_i$. A state is entangled if it is not separable.

A linear map $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$ is positive if Φ takes positive semi-definite operators on H_A to positive semi-definite operators on H_B. A map $\Phi \in L(\mathcal{A}_1, \mathcal{A}_2)$ is defined to be completely positive if $I_k \otimes \Phi : M_k \otimes \mathcal{A}_1 \to M_k \otimes \mathcal{A}_2$ is positive for all k, where I_k is the identity map on M_k. As pointed out by Kraus[42], a physical transformation of quantum systems should be completely positive, so such maps play a central role in quantum information theory.

If $V \in L(H_B, H_A)$, we denote by Ad_V the map in $L(\mathcal{A}_1, \mathcal{A}_2)$ given by

$$\operatorname{Ad}_V(X) = V^\dagger X V.$$

It is clear that Ad_V is a positive map, and in fact is completely positive since $I \otimes \mathrm{Ad}_V = \mathrm{Ad}_{I\otimes V}$. Every completely positive map Φ is a sum of such maps, $\Phi = \sum_i \mathrm{Ad}_{V_i}$. (This is often called a Kraus decomposition of Φ, cf. Ref. 43. A proof can be found in Refs. 8 and 42.)

Finally, we single out the following notion that will play an important role in our discussions.

Definition. An operator W in $\mathcal{A}_1 \otimes \mathcal{A}_2$ is *block positive* if $\langle W(x \otimes y), x \otimes y\rangle \geq 0$ for all x in H_A, y in H_B.

Correspondence of linear maps and operators

We review the Choi-Jamiołkowski isomorphism, which is an indispensable tool in working with positive and completely positive maps. We denote by E_{ij} the standard matrix units in M_n, i.e. $E_{ij} = e_i e_j^*$.

Definition. If Φ is a linear map from $\mathcal{A}_1$ to $\mathcal{A}_2$, then the Choi matrix C_Φ in $\mathcal{A}_1 \otimes \mathcal{A}_2$ is

$$C_\Phi = \sum_{ij} E_{ij} \otimes \Phi(E_{ij}).$$

If we define

$$P_+ = \sum_{ij} E_{ij} \otimes E_{ij}, \tag{1}$$

then $\frac{1}{m}P_+$ is the pure state associated with the maximally entangled vector $\psi_+ = \frac{1}{\sqrt{m}}\sum_i e_i \otimes e_i$, and $C_\Phi = (I \otimes \Phi)P_+$, where I is the identity on $\mathcal{A}_1$.

The map that takes Φ to C_Φ is readily seen to be a linear isomorphism from $L(\mathcal{A}_1, \mathcal{A}_2)$ to $\mathcal{A}_1 \otimes \mathcal{A}_2$, and is known as the Choi-Jamiołkowski isomorphism. It has the following properties. (Property (i) is due to Jamiołkowski[40] (who proved a slightly different but equivalent version), while (ii) is due to Choi[8]).

Theorem 1. *Let Φ be a linear map from $\mathcal{A}_1$ to $\mathcal{A}_2$.*

(i) Φ is positive iff C_Φ is block positive.

(ii) Φ is completely positive iff C_Φ is positive semi-definite.

For further discussion of this correspondence and related correspondences, see Refs. 48, 50, and 53.

Detecting entanglement

Entangled states are needed for most applications of quantum information theory, so it is important to be able to detect whether a given state is entangled or separable. We now review two means of entanglement detection: entanglement witnesses, and the positive maps criterion.

Entanglement witnesses

Two different necessary and sufficient conditions for separability were given by the Horodeckis[34]. For the first criterion, they applied the Hahn-Banach theorem to show that a state ρ on $H_A \otimes H_B$ is separable iff $\mathrm{tr}(\rho X) \geq 0$ for all block positive X. Thus if ρ is a state and W is block positive with $\mathrm{tr}(W\rho) < 0$, then ρ is entangled, so the observable W has in effect detected the entanglement of ρ. This led Terhal[74] to the following definition.

Definition. A block positive observable that detects entanglement of at least one state is an *entanglement witness*. Thus an entanglement witness W on $H_A \otimes H_B$ is a block positive operator that is not positive. We say W is normalized if $\mathrm{tr}\, W = 1$. (As shown by Lewenstein et al.[47], any nonzero block positive operator always has strictly positive trace, so we can always normalize a block positive operator.)

Theorem 2. *(Ref. 34) A state ρ on $H_A \otimes H_B$ is entangled iff $\mathrm{tr}\, \rho W < 0$ for some entanglement witness W. Thus every entangled state can be detected by an entanglement witness.*

Now we make use of the Choi-Jamiołkowski isomorphism. Note that if Φ is a positive map that is not completely positive, then C_Φ is block positive but not positive, so $\Phi \mapsto C_\Phi$ is a 1-1 correspondence of positive maps that are not completely positive with entanglement witnesses.

For an example, let the flip operator $V : \mathbb{C}^d \otimes \mathbb{C}^d \to \mathbb{C}^d \otimes \mathbb{C}^d$ be the linear operator satisfying $V(x \otimes y) = y \otimes x$. Then $\langle V(x \otimes y), (x \otimes y)\rangle = |\langle x, y\rangle|^2 \geq 0$, so V is block positive. The flip operator is an entanglement witness that gives a necessary and sufficient condition for detecting entanglement of the family of Werner states[76].

The positive maps criterion

A simple but very useful criterion for separability was proposed by Peres[54]. Let $t : \mathcal{A}_2 \to \mathcal{A}_2$ be the transpose map. If ρ is a separable state on $H_A \otimes H_B$, then $(I \otimes t)\rho$ will also be positive, and the property that $\rho^\Gamma = (I \otimes t)\rho \geq 0$ is called the positive partial transpose (PPT) property. A positive operator with positive partial transpose is called a PPT operator, and in particular a state with positive partial transpose is called a PPT state.

Earlier (before the notion of separability had been defined) Choi[9] raised the question of determining when an operator with the PPT property is a sum $\sum_i A_i \otimes B_i$ with $A_i \geq 0, B_i \geq 0$, and gave a 3×3 example where this is not the case.

The PPT criterion can be generalized by replacing the transpose map by any positive map. Let $\mathcal{A}_1 = L(H_A)$, $\mathcal{A}_2 = L(H_B)$, $\mathcal{A}_3 = L(H_C)$, and let $\Phi : \mathcal{A}_3 \to \mathcal{A}_2$ be a positive map. (Typically $H_C = H_A$ so $\mathcal{A}_3 = \mathcal{A}_1$, or $H_C = H_B$ so $\mathcal{A}_3 = \mathcal{A}_2$.) If ρ is a separable state on $H_A \otimes H_B$ then $(I \otimes \Phi^*)\rho \geq 0$. If this fails for some positive map Φ then ρ must be entangled.

Definition. Let $\mathcal{A}_1 = L(H_A)$, $\mathcal{A}_2 = L(H_B)$, $\mathcal{A}_3 = L(H_C)$, and let $\Phi : \mathcal{A}_3 \to \mathcal{A}_2$ be a positive map. If ρ is a state on $H_A \otimes H_B$ and if $(I \otimes \Phi^*)(\rho) \ngeq 0$, then we say that Φ *detects entanglement of* ρ.

The Horodeckis[34] showed that every entangled state can be detected by a positive map, by proving the following theorem.

Theorem 3. *(Positive Maps Criterion) A state ρ on $H_A \otimes H_B$ is separable iff for all positive maps $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$, $(I \otimes \Phi^*)\rho \geq 0$.*

Using results on decomposability of positive maps (discussed in the next section) and the positive maps criterion, the Horodeckis showed that the PPT property is a necessary and sufficient condition for separability in $M_2 \otimes M_2$, $M_2 \otimes M_3$, and $M_3 \otimes M_2$, but is not sufficient for $M_m \otimes M_n$ with $mn > 6$, cf. Ref. 34.

Horodecki, Smolin, Terhal, and Thapliyal[39] showed that the PPT property implies separability for any state of rank two or less. Thus if x is an entangled unit vector and P_x is the corresponding projection, it follows that P_x doesn't have the PPT property. Therefore $P_x^\Gamma \ngeq 0$, and since $P_x^\Gamma \geq 0$ on separable states, each P_x^Γ is an entanglement witness.

Let W be any entanglement witness in $\mathcal{A}_1 \otimes \mathcal{A}_2$, and $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$ the positive map such that $W = C_\Phi$. Generally Φ is a more powerful detector of entangled states than W in the sense that it detects every state detected by W and perhaps many more. Indeed, if C_Φ detects entanglement of a state ρ then

$$0 > \operatorname{tr}(C_\Phi \rho) = \operatorname{tr}((I \otimes \Phi)P_+)\rho = \operatorname{tr} P_+((I \otimes \Phi^*)\rho),$$

so Φ also detects entanglement of ρ. Furthermore, if X is any block positive operator then $W_X = (I \otimes \Phi)X$ is block positive, and all states detected by W_X are also detected by the positive map Φ. Thus Φ detects all states detected by the family W_X as X ranges over block positive operators.

Clearly the transpose map $t : M_n \to M_n$ detects precisely the non-PPT states on $M_m \otimes M_n$. For $m = n = 2$ the transpose map detects all entangled states, while this isn't true for the associated entanglement witness $C_t = V$ (where V is the flip map $V(x \otimes y) = y \otimes x$).

Decomposability of positive maps and entanglement witnesses

Definition. A positive map $\Phi : \mathcal{A}_2 \to \mathcal{A}_1$ is *decomposable* if it can be written in the form $\Phi = \Phi_1 + \Phi_2 \circ t$ where Φ_1, Φ_2 are completely positive. An operator $X \in \mathcal{A}_1 \otimes \mathcal{A}_2$ is decomposable if there are positive operators P, Q with $X = P + Q^\Gamma$.

From the definition of the Choi matrix, we have $C_{t\circ\Phi\circ t} = C_\Phi^t$. Thus $C_\Phi \geq 0$ iff $C_{t\circ\Phi\circ t} \geq 0$, so $t \circ \Phi \circ t$ is completely positive iff Φ is completely positive. Since $\Phi \circ t = t \circ (t \circ \Phi \circ t)$, it follows that decomposable maps can also be described as those of the form $\Phi_1 + t \circ \Phi_2$ for Φ_1, Φ_2 completely positive.

Decomposable operators are precisely the operators associated with decomposable positive maps under the Choi-Jamiołkowski isomorphism. To see this observe that

$$C_{\Phi_1 + t\circ\Phi_2} = C_{\Phi_1} + C_{t\circ\Phi_2} = C_{\Phi_1} + C_{\Phi_2}^\Gamma.$$

By results of Woronowicz[77] and Størmer[65], if $\dim H_A \dim H_B \leq 6$ all positive maps are decomposable, but this is not true in higher dimensions.

Examples of decomposable and indecomposable maps

The transpose map $t : M_d \to M_d$ is a positive map which is evidently decomposable. The reduction map $R : M_d \to M_d$ given by

$$R(\rho) = (\operatorname{tr} \rho)I - \rho$$

is a positive map defined by the Horodeckis[33]. By the positive map criterion, if ρ is separable then $(I \otimes R)\rho \geq 0$, and this is called the reduction criterion for separability. The corresponding entanglement witness is $C_R = I \otimes I - P_+$. Since $C_R^\Gamma = I \otimes I - V$, where V is the flip map, and $I \otimes I - V \geq 0$, then C_R is decomposable, and so the reduction map is decomposable.

The first explicit example of an indecomposable positive map was the Choi map on M_3, defined by

$$\Phi(X) = \begin{pmatrix} x_{11} + \mu x_{33} & -x_{12} & -x_{13} \\ -x_{21} & x_{22} + \mu x_{11} & -x_{23} \\ -x_{31} & -x_{32} & x_{33} + \mu x_{22} \end{pmatrix}$$

This was shown by Choi and Lam[10–12] to be indecomposable (and extremal in the cone of positive maps) by an argument involving the associated biquadratic form

$$F(x, y) = \langle \Phi(x^\dagger x)y, y \rangle \text{ for } x, y \in \mathbb{C}^m.$$

We will discuss in Theorem 5 below a more direct proof due to Størmer.

Breuer[5] and Hall[32] independently defined what are now called the Breuer-Hall maps Λ_d, on M_{2d} that generalize the reduction map. Let U be an antisymmetric unitary on $\mathbb{C}^{2d}$. Then

$$\Lambda_d^U(\rho) = \frac{1}{2d-2}((\operatorname{tr}\rho)I - \rho - U\rho^t U^\dagger),$$

and Breuer and Hall showed each map Λ_d^U is positive and indecomposable.

In Ref. 66 Størmer considered unital projections (positive maps P of M_d into itself such that $P^2 = P$ and $P(I) = I$), and described when they were completely positive or decomposable. This was used by Robertson to create the first example of an indecomposable positive map on M_4. He also showed that what is now called the Robertson map is extremal in the cone of positive maps. The Robertson map $\Phi : M_4 \to M_4$ is given by

$$\Phi(x_{ij}) = \begin{pmatrix} x_{33} + x_{44} & 0 & x_{13} + x_{42} & x_{14} - x_{32} \\ 0 & x_{33} + x_{44} & x_{23} - x_{41} & x_{24} + x_{31} \\ x_{31} + x_{24} & x_{32} - x_{14} & x_{11} + x_{22} & 0 \\ x_{41} - x_{23} & x_{42} + x_{13} & 0 & x_{11} + x_{22} \end{pmatrix}$$

Duality of cones

Let V_1, V_2 be finite dimensional real vector spaces with a pairing $\langle \cdot, \cdot \rangle$ (i.e., a bilinear form on $V_1 \otimes V_2$ such that $\langle x, y \rangle = 0$ for all $x \in V_1$ implies $y = 0$, and $\langle x, y \rangle = 0$ for all $y \in V_2$ implies $y = 0$.) One example of such a pairing is $\langle X, Y \rangle = \operatorname{tr} XY$ for X, Y Hermitian in $\mathcal{A}_1 \otimes \mathcal{A}_2$, which pairs the set of Hermitian operators $(\mathcal{A}_1 \otimes \mathcal{A}_2)_h$ with itself, and this will be the pairing understood unless otherwise mentioned.

A nonempty subset C of a real vector space V_1 is a *cone* if it is closed under multiplication by nonnegative scalars, and under sums. If we have a non-degenerate pairing $\langle \cdot, \cdot \rangle$ of V_1 and V_2, and if C is a cone in V_1 its dual cone is

$$C^* = \{Y \in V_2 \mid \langle X, Y \rangle \geq 0 \text{ for all } X \in C\}.$$

(This is the negative of the polar cone of C.) For a closed cone C, we have $C^{**} = C$, and if C_1, C_2 are closed cones,

$$(C_1 \cap C_2)^* = C_1 + C_2 \text{ and } (C_1 + C_2)^* = C_1^* \cap C_2^*.$$

We will see that duality of cones is useful in checking decomposability, and more generally in working with positive maps and block positive maps.

If K is any convex subset of a real vector space, then the set of non-negative multiples of elements of C is a cone, called the cone generated by K. We will make frequent reference to the cones generated by separable states and the cone generated by PPT states, and slightly abusing language we will refer to these as the cone of separable states and the cone of PPT states.

By the definition of block positive operators, the dual of the cone of separable states is the cone of block positive operators, and hence since the cone of separable states is closed, these cones are dual cones of each other. Decomposable operators and the cone of PPT states also are dual cones (see the next lemma). Each cone C of positive maps that corresponds under the Choi-Jamiołkowski isomorphism to one of the cones of decomposable, PPT, separable, positive, or block positive operators has the property that if Φ is in the cone C, and Ψ is completely positive, then $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are in the cone. Duality for such "mapping cones" was investigated by Størmer and Skowronek cf.[63,69,70].

Lemma 4. *The cone of PPT states in $\mathcal{A}_1 \otimes \mathcal{A}_2$ and the cone of decomposable operators are dual cones.*

Proof. Let $\mathcal{P}$ denote the positive cone. It is well known that this cone is self-dual, i.e. $\mathcal{P}^* = \mathcal{P}$. Recall that Γ denotes the partial transpose map. Since $\Gamma^* = \Gamma$, then $\mathcal{P}^\Gamma$ is also self-dual. Then

$$(\mathcal{P} \cap \mathcal{P}^\Gamma)^* = \mathcal{P}^* + (\mathcal{P}^\Gamma)^* = \mathcal{P} + \mathcal{P}^\Gamma.$$

The set of PPT states is $\mathcal{P} \cap \mathcal{P}^\Gamma$, and the set of decomposable operators is $\mathcal{P} + \mathcal{P}^\Gamma$, so the lemma follows. □

Størmer[67] gave the following test for decomposability of a positive map and applied it to show the Choi map is not decomposable.

Theorem 5. *A positive map $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$ is decomposable iff $I \otimes \Phi$ maps PPT operators to positive operators.*

Proof. Assume $\rho \in \mathcal{A}_1 \otimes \mathcal{A}_2$ is PPT, and $\Phi \in L(\mathcal{A}_1, \mathcal{A}_2)$ is decomposable, say $\Phi = \Phi_1 + \Phi_2 \circ t$ with Φ_1, Φ_2 completely positive, then

$$(I \otimes \Phi)\rho = (I \otimes \Phi_1)\rho + (I \otimes \Phi_2)((I \otimes t)(\rho)) \geq 0. \tag{2}$$

For the converse, see Ref. 67. □

Thus decomposable positive maps can't detect entanglement of PPT entangled states. Similarly, if $Q \geq 0$ and ρ is a PPT state, then $\langle Q^\Gamma, \rho \rangle = \langle Q, \rho^\Gamma \rangle \geq 0$, so decomposable entanglement witnesses can't detect entanglement of PPT states.

Optimal entanglement witnesses

For the sake of efficiency, one would like to use entanglement witnesses that detect as many entangled states as possible. If W is an entanglement witness, let $D_W = \{\rho \mid \mathrm{tr}(W\rho) < 0\}$ denote the set of entangled states detected by W. Lewenstein et al.[47] gave the following definition.

Definition. An entanglement witness W is *optimal* if W detects a maximal set of entangled states, i.e., if $D_W \subset D_{W_2}$ for an entanglement witness W_2 implies W_2 is a multiple of W.

There are other notions of optimality, e.g., the notion of an nd-optimal entanglement witness defined in Ref. 47 that involves maximality of the set of entangled PPT states detected by an entanglement witness. This is not the same as an optimal entanglement witness that happens to be indecomposable, as shown by Ha and Kye[28], and the latter is what we will mean when we use the term indecomposable optimal entanglement witness.

Lemma 6. *(Ref. 47) Let W_1, W_2 be entanglement witnesses. If $D_{W_1} = D_{W_2}$, then W_1 is a multiple of W_2.*

(The analogous statement for positive maps is not true. For example, transpose maps with respect to different orthonormal product bases each detect all entangled states on $M_2 \otimes M_2$.)

If W_1, W_2 are entanglement witnesses with $D_{W_1} \subset D_{W_2}$ and with W_2 not a multiple of W_1, we say W_2 is finer than W_1.

Lemma 7. *(Ref. 47) If W_1, W_2 are normalized entanglement witnesses such that W_2 is finer than W_1, then $W_1 = (1-\epsilon)W_2 + \epsilon P$, for some $0 < \epsilon < 1$ and $P \geq 0$.*

It follows that an entanglement witness W (not necessarily normalized) is optimal iff it cannot be written as a convex combination of an entanglement witness W_2 and a positive (nonzero) operator. Equivalently W is optimal iff there is no positive operator P such that $W - P$ is block positive.

Definition. A positive map Φ that is not completely positive is optimal if the corresponding entanglement witness is optimal. (This is equivalent to there being no nonzero completely positive map Ψ with $\Phi \geq \Psi$.)

Note that the set of states detected by an optimal positive map isn't necessarily maximal among sets detected by positive maps. For example, if the reduction map detects entanglement of a state, then so does the transpose map, and in $M_n \otimes M_n$ for $n \geq 3$ there are states detected by the transpose map but not by the reduction map, cf. Ref. 33. Thus the set of entangled states detected by the reduction map is a proper subset of the set of entangled states detected by the transpose map. However, both are optimal positive maps (as we will see later).

We now discuss the close connection between optimality of entanglement witnesses and the facial structure of the cone of block positive operators (or of the compact convex set of normalized block positive operators), starting with extremal operators.

Definition. $\mathcal{BP}$ is the cone of block positive operators on $H_A \otimes H_B$. We write $\mathcal{BP}_1$ for the compact convex set of normalized block positive operators.

Arguments involving the cone $\mathcal{BP}$ often can be rephrased in terms of the compact convex set $\mathcal{BP}_1$. There isn't as natural a way to normalize positive maps.

Definition. Let C be a cone in a real vector space V. A nonzero element $x \in C$ is *extremal* if whenever x is written as a convex combination of $x_1, x_2 \in C$, then each of x_1, x_2 is a multiple of x. (We will define faces of convex sets later and see that x is extremal in a cone C iff the ray $\{\lambda x \mid 0 \leq \lambda \in \mathbb{R}\}$ is a face of C.)

Definition. A block positive operator W is extremal if it is extremal in the cone $\mathcal{BP}$. (This is equivalent to $\mathrm{tr}(W)^{-1}W$ being an extreme point of the set $\mathcal{BP}_1$ of normalized block positive operators.)

An extremal entanglement witness is defined to be an entanglement witness that is an extremal block positive operator.

A positive map $\Phi \in L(\mathcal{A}_1, \mathcal{A}_2)$ is extremal if it is extremal in the cone of positive maps. This is equivalent to C_Φ being extremal in the cone $\mathcal{BP}$.

Note that the set of block positive observables is convex, while the set of entanglement witnesses is not. For example, if $P_1, \ldots, P_4$ are the four Bell states, then each P_i^Γ is an entanglement witness. Then $\frac{1}{4}\sum_i P_i^\Gamma = \frac{1}{4}(I \otimes I)$ is block positive but detects no entangled state, hence is not an entanglement witness.

By definition every extremal entanglement witness is an extremal block positive operator, but there are extremal block positive operators that are positive and thus detect no entangled states, hence are not entanglement witnesses. For example, if $V \in L(H_B, H_A)$ and $\mathrm{Ad}_V(X) = V^\dagger X V$, then Ad_V is a completely positive map that is extremal both among completely positive maps and among positive maps, see Thm. 3.5 in Ref. 70. Then the Choi-Jamiołkowski isomorphism carries Ad_V to an extremal positive operator in $\mathcal{A}_1 \otimes \mathcal{A}_2$ that is also an extremal block positive operator, but is not an entanglement witness.

The following is one way to prove an entanglement witness is optimal.

Lemma 8. *If W is an extremal entanglement witness, then W is optimal.*

Proof. This follows at once from Lemma 7.

□

The following property is one of the most common ways used to prove optimality.

Definition. For an entanglement witness W, let Z_W be the set of product vectors $x \otimes y$ in $H_A \otimes H_B$ such that $\langle W(x \otimes y), x \otimes y \rangle = 0$. An entanglement witness has the *spanning property* if the linear span of Z_W is all of $H_A \otimes H_B$.

Lemma 9. *(Ref. 47) If an entanglement witness W has the spanning property, then W is optimal.*

Thus both the spanning property and extremality imply that an entanglement witness is optimal. These properties are independent. The indecomposable positive map described by Choi[10,11] is extremal[12] but doesn't have the spanning property (see the papers of Korbicz, Almeida, Bae, and Lewenstein[41], and of Kye[46]). On the other hand, examples are given by Ha and Kye[28], and by Chruściński and Pytel[14], of positive maps with the spanning property that are not extremal. Finally, there are examples of optimal entanglement witnesses that are nether extremal nor spanning. Positive maps in a family defined by Qi and Hou[56] were shown to to be indecomposable optimal entanglement witnesses not having the spanning property in Ref. 57, and then some in that family were shown not to be extremal by Ha and Yu[31].

The two best known examples of optimal positive maps are the transpose map and the reduction map. Both are decomposable and both have been used in well known tests for separability via the positive maps criterion. It is straightforward to check that the transpose map is extremal among positive maps, and is not completely positive, hence is optimal. The reduction map is extremal if $d = 2$ but not for $d > 2$. It has the spanning property in all dimensions, see Ref. 14, hence is optimal.

Remarkably, Augusiak, Tura, and Lewenstein[2] showed that in $M_2 \otimes M_n$, for a decomposable entanglement witness W, the following are equivalent: (i) W is optimal (ii) $W = Q^\Gamma$ where the range of Q is completely entangled, i.e. has no product vectors (iii) W has the spanning property. So in particular optimality implies the spanning property in $M_2 \otimes M_n$. (For 3×3 systems one needs to add to (ii) the assumption that the rank of Q is one or two.)

Facial structure

We have seen above that extremality implies optimality for entanglement witnesses, but is not necessary. Necessary and sufficient conditions for optimality can be given by making use of the facial structure of $\mathcal{BP}$.

Definition. If K is a convex set, a convex subset F of K is a *face* of K if whenever a mixture (convex combination) $t\sigma + (1-t)\tau$ is in F with $\sigma, \tau \in K$, then $\sigma, \tau \in F$. (In other words, F is a face if any line segment in K whose interior meets F is contained in F.) A proper face of K is a nonempty face that is not all of K.

Thus extreme points of K are the faces of K that consist of a single point. If $\rho \in K$, then $\mathrm{face}_K(\rho)$ is the face of K consisting of all points on line segments whose interior contains ρ. This is the minimal face containing ρ in the sense that it is contained in any face of K that contains ρ. If C is a cone, then points W in C are extremal iff the ray $\{\lambda W \mid 0 \leq \lambda \in \mathbb{R}\}$ they generate is a face of C.

If K is a compact convex set in a finite dimensional space, then the boundary of K is the union of the proper faces of K, and is the disjoint union of their relative interiors, cf. Thm. 2.1.2 in Ref. 62. If F is a proper face of K then $\dim F < \dim K$. The face generated by $\rho \in K$ will be all of K iff ρ is a (relative) interior point of K, and is a proper face of K iff ρ is a (relative) boundary point of K.

Exposed faces

Recall that function σ on a convex set is affine if σ preserves convex combinations: $\sigma(tX + (1-t)Y) = t\sigma(X) + (1-t)\sigma(Y)$ for all X, Y in the convex set and for $0 \leq t \leq 1$. A face F of a finite dimensional convex set K is said to be *exposed* if there is an affine functional on K which is nonnegative on K and whose zero set on K is F. (An exposed face of K can be visualized as the result of translating a hyperplane not meeting K until it first touches K; the intersection is an exposed face of K.)

All faces of some convex sets are exposed, for example, all faces of polytopes are exposed, and all faces of the positive cone of M_n are exposed. (It has long been known that faces of the positive cone are the sets of the form $F_P = \{\rho \geq 0 \mid \mathrm{tr}(\rho P) = 0\}$ for projections P, see for example Refs. 3, 22, and 55). Therefore, some authors find it convenient to define "face" to be what we have called an exposed face. However, there are faces of convex sets of interest in quantum information that are not exposed. For example Eom and Kye[23] showed that the nondecomposable positive map described by Choi[11] is extremal but is not exposed in the cone of positive maps. A simple geometric illustration of a convex set with non-exposed extreme points is the convex hull of a circular disk and a point outside the disk. Note that being exposed depends on the context: in the example just given, the two non-exposed points are exposed with respect to the facial line segment they belong to.

Let V_1, V_2 be spaces with a pairing $\langle \cdot, \cdot \rangle$, and let C be a closed cone in V_1 with dual cone C^* in V_2. If E is a subset of the cone C, then we write $E^\diamond = \{Y \in C^* \mid \langle X, Y \rangle = 0 \text{ for all } X \in E\}$. Then $F \subset C$ is an exposed face of C iff $F = F^{\diamond\diamond}$. For any subset E of C, $E^{\diamond\diamond}$ will be an exposed face of C, and will be contained in any exposed face that contains E. We write $\mathrm{expface}(E)$ for the minimal exposed face containing E, i.e. $\mathrm{expface}(E) = E^{\diamond\diamond}$, and call $\mathrm{expface}(E)$ the exposed face generated by E.

With slight abuse of language, a positive map is said to be exposed in the cone of positive maps if the ray generated by that map is an exposed face of the cone of positive maps. Similarly a block positive operator is said to be exposed if the ray generated by that witness is an exposed face of the cone $\mathcal{BP}$ of block positive operators. (This is equivalent to the normalized operator being an exposed point of the convex set $\mathcal{BP}_1$.)

Optimality and facial structure

We can rephrase Lemma 7 as follows.

Lemma 10. *An entanglement witness W is optimal iff $\mathrm{face}_{\mathcal{BP}} W$ contains no positive elements other than 0.*

Kye in Prop. 8.4 of Ref. 44 pointed out the following facial characterization of the spanning property.

Lemma 11. *An entanglement witness $W \in M_m \otimes M_n$ has the spanning property iff the exposed face of $\mathcal{BP}$ generated by W contains no positive operator.*

It follows that any exposed entanglement witness has the spanning property. Note that the partial transpose map Γ leaves invariant each of the cones of PPT, decomposable, and separable operators. Since $\Gamma^* = \Gamma$, then Γ also leaves the cone of block positive operators invariant. Thus Γ is an affine automorphism of each of these cones, hence takes faces to faces and exposed faces to exposed faces. In particular, if W is an entanglement witness such that $W^\Gamma \not\geq 0$, then W is extremal iff W^Γ is extremal, and W is exposed iff W^Γ is exposed. Thus if W is an indecomposable entanglement witness, these equivalences hold.

The following result is a slight variation of Lemma 22 in Sarbicki[60].

Lemma 12. *If W is an optimal entanglement witness, then every entanglement witness in* $\operatorname{face}_{\mathcal{BP}} W$ *is optimal.*

Proof. If W is optimal by Lemma 10 $\operatorname{face}_{\mathcal{BP}} W$ contains no nonzero $P \geq 0$. Let $W' \in \operatorname{face}_{\mathcal{BP}} W$, then $\operatorname{face}_{\mathcal{BP}} W' \subset \operatorname{face}_{\mathcal{BP}} W$ so $\operatorname{face}_{\mathcal{BP}} W'$ contains no nonzero positive operator. Thus by Lemma 10, W' is optimal. □

Similarly, if an entanglement witness W has the spanning property, every entanglement witness in the exposed face of $\mathcal{BP}$ generated by W also has the spanning property.

From lemmas 10 and 12 it follows that the set of optimal entanglement witnesses is the union of the faces of $\mathcal{BP}$ that contain no positive nonzero operator, and all such faces are contained in the boundary of $\mathcal{BP}$. However, there are operators on the boundary of $\mathcal{BP}$ that are not optimal.

The following two results make more explicit a sense in which for detecting entanglement we can rely on optimal (or even extremal or exposed) entanglement witnesses.

Corollary 13. *(Refs. 16, 25, and 64) If ρ is an entangled state, then there is an extremal (hence optimal) entanglement witness that detects ρ. The witness can be chosen to be exposed.*

Proof. If W is an entanglement witness that detects ρ, we can assume without loss of generality that W is normalized. We can express W as a convex combination of extreme points of $\mathcal{BP}_1$, operators, at least one of which must also detect ρ. By Strasziewicz' Theorem[58], the exposed points of a compact convex set are dense in the set of extreme points, so there is an exposed witness that detects ρ. □

Thus for purposes of entanglement detection, we could just work with extremal, even exposed, entanglement witnesses. Similarly, one could restrict detection by positive maps to exposed positive maps.

Lewenstein et al.[47] showed that every entanglement witness W can be optimized, i.e., there is an optimal entanglement witness that is finer than W. In fact, that optimal entanglement witness can be chosen to be extremal.

Corollary 14. *If W is a non-optimal entanglement witness, then there is an extremal (hence optimal) entanglement witness that is finer than W.*

Proof. In Ref. 47 an algorithm is sketched to optimize an entanglement witness, i.e., to find an optimal entanglement witness that refines the given one. With slight modification, this gives an extremal entanglement witness. We sketch the idea.

We may assume W is normalized, and show there is a normalized extremal entanglement witness that is finer than W. Note for a normalized entanglement witness extremality is the same as being an extreme point of $\mathcal{BP}_1$. We use induction on $\dim \text{face}_{\mathcal{BP}_1} W$. If this dimension is 0 then W is extremal, hence optimal, so there is nothing to prove. Suppose the corollary holds for $\dim \text{face}_{\mathcal{BP}} W < k$, and let W be a non-optimal entanglement witness with $\dim \text{face}_{\mathcal{BP}} W = k$.

If W is not optimal, by Lemma 7 we can find $P \geq 0$ and an entanglement witness W_1 such that W is in the interior of the line segment $[P, W_1]$. Since $\mathcal{BP}_1$ and $\text{face}_{\mathcal{BP}_1} W$ are compact, the extension of this line segment must meet the (relative) boundary of $\text{face}_{\mathcal{BP}_1}$, so we can choose W_1 to be in this relative boundary, say with $W = tW_1 + (1-t)P$. Then $\dim \text{face}_{\mathcal{BP}_1} W_1 < \dim \text{face}_{\mathcal{BP}_1} W$. Now by the induction hypothesis there is an extremal entanglement witness W' finer than W_1, and thus there exists $0 < s < 1$ and nonzero $Q \geq 0$ such that $W_1 = sW' + (1-s)Q$. Combining gives

$$W = tW_1 + (1-t)P = t(sW' + (1-s)Q) + (1-t)P = tsW' + t(1-s)Q + (1-t)P,$$

and then W' is finer than W. $\square$

Examples of exposed positive maps and entanglement witnesses

If V is a linear map from H_B to H_A, Størmer[70] showed that the maps Ad_V and $\text{Ad}_V \circ t$ are extremal positive maps. It was shown by Marciniak[49] that such maps are exposed in the cone of positive maps.

Theorem 15. *If x is any unit vector in $H_A \otimes H_B$, then the associated projection P_x is exposed in the cone of block positive operators. Thus if x is entangled, then P_x^Γ will be an exposed entanglement witness.*

Proof. We first show that $P_x = C_{\text{Ad}_V}$ for some V. Indeed, since P_x is extremal in the cone of positive operators, then the corresponding map under the Choi-Jamiołkowski isomorphism is an extremal completely positive map, hence must be of the form C_{Ad_V}. By Marciniak's result, C_{Ad_V} is exposed not only in the cone of completely positive maps, but also in the full cone of positive maps. It follows that P_x is exposed in the cone of block positive operators.

The partial transpose map Γ is an affine isomorphism on the cone of block positive operators, so P_x^Γ is also an exposed block positive operator. If x is entangled, we saw before that P_x^Γ is an entanglement witness.

□

Chruściński and Sarbicki[16] developed a sufficient criterion for a positive map to be exposed, and then applied this in Ref. 17 to show the Breuer-Hall maps are exposed. In Ref. 61 they showed that the Robertson map and some higher dimensional generalizations are exposed.

Of course, exposed entanglement witnesses are also extremal, hence optimal. As the next result indicates, if W is exposed and indecomposable, the same is true of W^Γ. Thus exposed entanglement witnesses are a rich source of optimal entanglement witnesses and optimal positive maps.

Theorem 16. *Let W be an exposed entanglement witness.*

(i) If W is decomposable, then $W = P_x^\Gamma$ for some entangled vector x.

(ii) If W is indecomposable, then W^Γ is also an exposed (indecomposable) entanglement witness.

Proof. (i) Write $W = P + Q^\Gamma$ where $P, Q \geq 0$. Since W is an entanglement witness,then $Q \neq 0$. Since W is extremal then $P = 0$, so $W = Q^\Gamma$. Then $W^\Gamma = Q$ will also be extremal in $\mathcal{BP}$, hence also extremal among positive operators. Thus $Q = P_x$ for some x. Then $W = P_x^\Gamma$. Here since W is not positive, then x can't be a product vector, hence is entangled.

(ii) Since W is exposed, then W^Γ also will be exposed in $\mathcal{BP}$. Since W is by assumption indecomposable, then $W^\Gamma \not\geq 0$, so W^Γ is an entanglement witness.

□

The structural physical approximation

As discussed previously, by virtue of the positive maps criterion for separability, positive maps are able to detect all entangled states. However, it is only completely positive maps that are physically realizable.

This led P. Horodecki[37], see also Ref. 38, to introduce the notion of the structural physical approximation of a positive map Φ, which is (up to a scalar multiple) a completely positive mixture of Φ and the completely depolarizing map, defined more precisely below. The idea is that on one hand $\mathrm{SPA}(\Phi)$ can be physically implemented, and on the other hand it is closely related to Φ and hence can be used for many of the same purposes, including entanglement detection, as we will discuss later after defining the SPA.

For a positive map Φ to be physically implementable it needs to be completely positive, but also can't increase trace. However, the latter property can always be accomplished by scaling the operator, replacing Φ by $\lambda^{-1}\Phi$ where λ is the maximum value of $\Phi(\rho)$ for states ρ. Hereafter we will assume this scaling has taken place, so that Φ is either trace preserving or trace non-increasing.

Definition. The completely depolarizing map $D : \mathcal{A}_1 \to \mathcal{A}_2$ is given by $D(X) = \mathrm{tr}(X) I_B/d_B$ where $d_B = \mathrm{tr}\, I_B$. (In other words, the completely depolarizing map transforms every state on H_A to the maximally mixed state on H_B.)

Definition. If $\Phi \in L(\mathcal{A}_1, \mathcal{A}_2)$ is a map that takes Hermitian operators to Hermitian operators, let t_* be the minimum value of t such that $(1-t)\Phi + tD$ is completely positive. (Since $C_{(1-t)\Phi+tD} = (1-t)C_\Phi + tC_D = (1-t)C_\Phi + tI \otimes I/d_B$, such numbers t always exist.) We define $\mathrm{SPA}(\Phi) = (1-t_*)\Phi + t_* D$.

Since a positive map Φ is completely positive iff the associated entanglement witness C_Φ is positive semi-definite, the following is the natural definition of the structural physical approximation for entanglement witnesses.

Definition. If W is any Hermitian operator on $H_A \otimes H_B$ with $\mathrm{tr}\, W = 1$, let t_* be the minimum value of t such that $(1-t)W + tI \otimes I/d_A d_B \geq 0$. The *structural physical approximation* of W is $\mathrm{SPA}(W) = (1-t_*)W + t_* I \otimes I/d_A d_B$. If $\mathrm{tr}\, W$ is nonzero but not equal to 1, we define $\mathrm{SPA}(W) = \mathrm{SPA}(W/\mathrm{tr}\, W)$.

One reason for the choice of the completely depolarizing map in constructing the SPA of a positive map is that it can be interpreted as adding a minimal amount of "white noise", cf. Ref. 41. Another virtue (discussed more in a moment) is that from $\mathrm{SPA}(\Phi)(\rho)$ one can recover very useful information about $\Phi(\rho)$. Finally, for entanglement witnesses adding a multiple of the identity has readily identifiable effects on expectation values.

For further motivation, following Horodecki and Ekert[38] we illustrate how the structural physical approximation could be used in entanglement testing, and in particular how the effects of mixing in the completely depolarizing map can be compensated for. To test entanglement of a state ρ, we want to test whether $(I\otimes\Phi)\rho \geq 0$ for a particular positive map Φ. Let $\Psi = I\otimes\Phi$ and let ρ be a state. Let $\mathrm{SPA}(\Psi) = (1-\lambda)\Psi+\lambda D$. Then to test positivity of $\Psi(\rho)$ we measure the spectrum of $\mathrm{SPA}(\Psi)(\rho) = ((1-\lambda)\Psi+\lambda D)\rho = (1-\lambda)\Psi(\rho)+\lambda I_B/d_B$, which is

$$\mathrm{spec}(\mathrm{SPA}(\Psi)(\rho)) = (1-\lambda)\,\mathrm{spec}(\Psi(\rho)) + \lambda/d_B.$$

Thus from the spectrum of $\mathrm{SPA}(\Psi)(\rho)$ and the scalar λ we can recover the spectrum of $\Psi(\rho) = (I\otimes\Phi)\rho$, and hence test if $(I\otimes\Phi)\rho$ is positive.

Since $\mathrm{SPA}(I\otimes\Phi)$ is completely positive, it can be implemented experimentally. Note that in this case we are making use of the SPA for a map $I\otimes\Phi$ that is not necessarily positive. Here $I\otimes\mathrm{SPA}(\Phi)$ is not the same as $\mathrm{SPA}(I\otimes\Phi)$, and it is really the latter that provides a physically implementable test for entanglement.

The SPA conjecture

The notion of an entanglement breaking map was investigated by Horodecki, Shor, and Ruskai[35].

Definition. A positive map $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$ is *entanglement breaking* if $I\otimes\Phi$ maps every state to a multiple of a separable state.

This is equivalent to $C_\Phi = (I\otimes\Phi)P_+$ being separable, cf. Ref. 35. Such maps are sometimes called superpositive maps, because of the property that when composed with any positive map, they remain completely positive.

As shown in Ref. 35, trace preserving entanglement breaking maps can also be characterized as those Φ which can be represented in Holevo form

$$\Phi(\rho) = \sum_k \mathrm{tr}(F_k\rho)\rho_k$$

where each F_k is positive and each ρ_k is a state and $\sum_k F_k = I$. This can be interpreted as a combination of a generalized measurement (corresponding to the F_k), followed by generating the state ρ_k if the measurement result was that associated with F_k. If Φ is trace non-increasing and entanglement breaking, then in this representation $\sum_k F_k \leq I$, and one can interpret this as a measurement where the state is discarded after the measurement if the outcome corresponds to none of the F_k.

The following conjecture was posed by Korbicz et al.[41].

SPA Conjecture for positive maps If $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$ is an optimal positive map, then $SPA(\Phi)$ is entanglement breaking.

Here is the equivalent conjecture phrased in terms of entanglement witnesses.

SPA Conjecture for entanglement witnesses If W is an optimal entanglement witness, then $SPA(W)$ is separable.

One challenging part of investigating this conjecture is that in $M_m \otimes M_n$ for $mn > 6$, no simple necessary and sufficient test of separability is known, other than for special families of states.

EXAMPLES SUPPORTING THE SPA CONJECTURE

The SPA conjecture when formulated by Korbicz et al.[41] was supported by quite a few examples in the original article, and we'll start by discussing some of those.

Theorem 17. *The SPA conjecture holds for positive maps $M_m \to M_n$ for $mn \leq 6$.*

Proof. All positive maps Φ on $M_m \otimes M_n$ with $mn \leq 6$ are decomposable, so the associated entanglement witness $W = C_\Phi$ will also be decomposable, say $W = P + Q^\Gamma$ for $P, Q \geq 0$. We may assume $\operatorname{tr} W = 1$. If W is optimal then $P = 0$. If $\mathrm{SPA}(W) = (1-t)W + tI \otimes I = (1-t)Q^\Gamma + tI \otimes I$, then $\mathrm{SPA}(W)$ is PPT. In the given dimensions, as discussed previously, PPT implies separability, so $\mathrm{SPA}(W)$ is separable. $\square$

Korbicz et al. show that the transpose map and reduction map each satisfy the SPA conjecture. They also show partial transposition has an entanglement breaking SPA if $d_A \geq d_B$. For $M_2 \otimes M_2$ the fact that $\mathrm{SPA}(I \otimes t)$ is entanglement breaking was proven earlier by Fiurásek[24] (though not using that terminology).

Theorem 18. *(Ref. 1) If $\psi \in \mathbb{C}^m \otimes \mathbb{C}^n$ is entangled, and P_ψ is the associated rank one projection, then $W = P_\psi^\Gamma$ is an optimal entanglement witness satisfying the* SPA *conjecture.*

Proof. We have seen above that W is an exposed, hence extremal, hence optimal entanglement witness. The authors show SPA(W) is separable by expressing W as a convex combination of explicit product states. □

Korbicz et al. show various examples of indecomposable positive maps are optimal and satisfy the SPA conjecture. Their examples include the Choi map on M_3, the Breuer-Hall family of maps on M_{2n}, and certain entanglement witnesses and associated positive maps built from unextendable product bases.

Chruściński and coauthors[13–15,78] defined a variety of generalizations on M_{2n} of the Robertson and Breuer-Hall maps and showed that these maps satisfy the SPA conjecture (including the Robertson map as a special case).

The following family of maps, defined by Cho, Kye, and Lee[7], generalize the Choi map.

Definition. Let a, b, c be nonnegative real numbers. Then the generalized Choi map $\Phi[a, b, c] : M_3 \to M_3$ is defined by

$$\Phi[a,b,c](X) = \begin{pmatrix} ax_{11} + bx_{22} + cx_{33} & -x_{12} & -x_{13} \\ -x_{21} & cx_{11} + ax_{22} + bx_{33} & -x_{23} \\ -x_{31} & -x_{32} & bx_{11} + cx_{22} + ax_{33} \end{pmatrix} \qquad (3)$$

where $X = (x_{ij})$.

Here $\Phi[1, 0, \mu]$ with $\mu \geq 1$ is the original Choi map, and $\Phi[0, 1, 1]$ is the reduction map on M_3.

Theorem 19. *(Ref. 7)*

(i) $\Phi[a, b, c]$ *is completely positive iff* $a \geq 2$ *and copositive iff* $bc \geq 1$.

(ii) $\Phi[a, b, c]$ *is positive iff*

$$a + b + c \geq 2 \text{ and } 0 \leq a \leq 1 \Rightarrow bc \geq (1 - a)^2$$

(iii) $\Phi[a, b, c]$ *is decomposable iff*

$$0 \leq a \leq 2 \Rightarrow bc \geq \left(\frac{2 - a}{2}\right)^2$$

The following results (i) of Ha and Kye[25,26], and (ii) of Chruściński and Wudarski[20] provided additional examples for the SPA conjecture was known to hold.

Theorem 20. *Let $0 < a < 1, a+b+c = 2, bc = (1-a)^2$. Then*

(i) $\Phi[a,b,c]$ is an exposed (hence optimal) positive map and is indecomposable.

(ii) If also $2b+c \leq 1$ and $2c+b \leq 1$, then $\mathrm{SPA}(\Phi[a,b,c])$ is entanglement breaking.

Qi and Hou[56] defined a generalization $\Phi^{n,k}$ of the Choi map to M_n for $n \geq 3$. In Ref. 57 they show for $1 \leq k \leq n-1$ with $k \neq n/2$ these maps are indecomposable and optimal and have entanglement breaking SPA.

The Choi maps were generalized to indecomposable maps in higher dimensions by Tanahashi and Tomiyama[73] and Osaka[51,52]. Augusiak, Bae, Czekaj, and Lewenstein[1] verified the SPA conjecture for these maps. They also formulated a version of the SPA conjecture for the continuous context, and verified the conjecture in some cases for that version of the SPA conjecture.

Augusiak et al.[1] also investigate variations on the structural positive approximation involving mixing the original map with the least needed proportion of an entanglement breaking map other than the completely depolarizing map. They show that in some cases for optimal positive maps this does not give an entanglement breaking map, but that for every positive map there is at least one entanglement breaking map for which the associated SPA constructed with that EB map is entanglement breaking.

In summary, a large variety of positive maps were found to satisfy the SPA conjecture.

HA AND KYE'S DISPROOF OF THE SPA CONJECTURE

To simplify some calculations in the next proof, the following variation of the SPA of an entanglement witness will be useful.

Definition. Let W be any Hermitian operator on $H_A \otimes H_B$. If $W \ngeq 0$, we define λ_W to be the number such that $-\lambda_W$ is the minimal negative eigenvalue of W. If $W \geq 0$ we define $\lambda_W = 0$.

Definition. If W is any Hermitian operator on $H_A \otimes H_B$, we define $\mathrm{SPA}_0(W) = W + \lambda_W I \otimes I$.

A straightforward calculation shows that $\mathrm{SPA}(W)$ is a multiple of $\mathrm{SPA}_0(W)$. Note that for any $\alpha > 0$, since $\lambda_{\alpha W} = \alpha \lambda_W$, then

$$\mathrm{SPA}_0(\alpha W) = \alpha W + \lambda_{\alpha W} I \otimes I = \alpha(W + \lambda_W I \otimes I) = \alpha\, \mathrm{SPA}_0(W).$$

Since λ_W depends continuously on W, then $\mathrm{SPA}(W)$ and $\mathrm{SPA}_0(W)$ are continuous functions of W.

The following (with somewhat different notation and terminology) is a central observation of Ha and Kye[29], and was stated later by Wang and Long[75] in the form given here.

Theorem 21. *Let W be an observable (a Hermitian operator) on $H_A \otimes H_B$.*

(i) If $\lambda_W < \lambda_{W^\Gamma}$, then $\mathrm{SPA}(W)$ is not PPT.

(ii) $\lambda_W > \lambda_{W^\Gamma}$, then $\mathrm{SPA}(W^\Gamma)$ is not PPT.

(iii) If $\lambda_W = \lambda_{W^\Gamma}$, then $\mathrm{SPA}(W)$ and $\mathrm{SPA}(W^\Gamma)$ are PPT, and $\mathrm{SPA}(W^\Gamma) = \mathrm{SPA}(W)^\Gamma$.

Ha and Kye describe the conditions above by saying that an entanglement witness W is of *positive type* if $\lambda_W \leq \lambda_{W^\Gamma}$, of *copositive type* if $\lambda_W \geq \lambda_{W^\Gamma}$, and of *PPT type* if $\lambda_W = \lambda_{W^\Gamma}$.

Proof. (i) Suppose $\mathrm{SPA}(W)$ is PPT. Then $\mathrm{SPA}_0(W) = W + \lambda_W I \otimes I$ is PPT. Therefore $W^\Gamma + \lambda_W I \otimes I \geq 0$, and so $\lambda_{W^\Gamma} \leq \lambda_W$. Thus if $\lambda_W < \lambda_{W^\Gamma}$, then $\mathrm{SPA}(W)$ is not PPT.

(ii) This follows by replacing W by W^Γ in (i).

(iii) For all W by definition $\mathrm{SPA}(W) \geq 0$, so $\mathrm{SPA}_0(W) \geq 0$. Since by assumption $\lambda_W = \lambda_{W^\Gamma}$, then

$$\mathrm{SPA}_0(W)^\Gamma = (W + \lambda_W I \otimes I)^\Gamma = W^\Gamma + \lambda_W I \otimes I = W^\Gamma + \lambda_{W^\Gamma} I \otimes I = \mathrm{SPA}_0(W^\Gamma) \geq 0.$$

Now $\mathrm{SPA}(W^\Gamma) = \mathrm{SPA}(W)^\Gamma$ follows. □

Thus if there is an optimal entanglement witness W such that (i) holds, then $\mathrm{SPA}(W)$ is not separable. If W is an entanglement witness with W^Γ optimal and with (ii) holding, then $\mathrm{SPA}(W^\Gamma)$ is not separable. In either case, this would disprove the SPA conjecture.

Finally, if W and W^Γ are both optimal entanglement witnesses such that $\lambda_W \neq \lambda_{W^\Gamma}$, then one or the other of W and W^Γ would be counterexamples to the SPA conjecture. If (iii) holds for W, we know that $\mathrm{SPA}(W)$ and $\mathrm{SPA}(W^\Gamma)$ are PPT, but whether they are separable would remain unresolved.

The family of generalized Choi maps $\Phi[a, b, c]$ defined by Cho, Kye, Lee, was generalized further by Ha and Kye[27] to a family $\Phi[a, b, c, \theta]$ described below. Then in Ref. 29 Ha and Kye find parameters such that $\Phi[a, b, c, \theta]$ is an optimal positive map satisfying case (iii) of Theorem 21 (for $W = C_{\Phi[a,b,c,\theta]}$). Then using previous results of Kye and Osaka[45] they show that $W[a, b, c, \theta]$ is not separable, so $\Phi[a, b, c, \theta]$ is not entanglement breaking, disproving the SPA conjecture. We now summarize their argument.

Definition. Let a, b, c be nonnegative real numbers, and $-\pi \leq \theta \leq \pi$. Then the generalized Choi map $\Phi[a, b, c, \theta] : M_3 \to M_3$ is defined by

$$\Phi[a,b,c,\theta](X) = \begin{pmatrix} ax_{11} + bx_{22} + cx_{33} & -e^{i\theta}x_{12} & -e^{-i\theta}x_{13} \\ -e^{-i\theta}x_{21} & cx_{11} + ax_{22} + bx_{33} & -e^{i\theta}x_{23} \\ -e^{i\theta}x_{31} & -e^{-i\theta}x_{32} & bx_{11} + cx_{22} + ax_{33} \end{pmatrix}$$

where $X = (x_{ij})$.

Here is the Choi matrix for the Choi maps $\Phi[a, b, c, \theta]$ (where for greater readability, zeros are represented by dots):

$$W[a,b,c,\theta] = \begin{pmatrix} a & \cdot & \cdot & \cdot & -e^{i\theta} & \cdot & \cdot & \cdot & -e^{-i\theta} \\ \cdot & c & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot \\ -e^{-i\theta} & \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot & -e^{i\theta} \\ \cdot & \cdot & \cdot & \cdot & \cdot & c & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b & \cdot \\ -e^{i\theta} & \cdot & \cdot & \cdot & -e^{-i\theta} & \cdot & \cdot & \cdot & a \end{pmatrix}$$

The following parameter will play a key role in the results that follow. For $-\pi \leq \theta \leq \pi$, define

$$p_\theta = 2 \max\{\cos(\theta + \frac{2}{3}\pi), \cos\theta, \cos(\theta - \frac{2}{3}\pi)\}$$

Note that $0 \leq p_\theta \leq 2$, with $p_\theta = 1$ iff $\theta = \pm\pi/3, \pm 2\pi/3$ and $p_\theta = 2$ iff $\theta = 0, \pm 2\pi/3$.

Ha and Kye[27] characterized positivity and complete positivity of these maps as described next.

Theorem 22. *Let $\Phi = \Phi[a, b, c, \theta]$ and $W = C_\Phi$.*

(i) Φ is completely positive (equivalently W is positive) iff $a \geq p_\theta$, and W^Γ is positive iff $bc \geq 1$.

(ii) Φ is a positive map (equivalently W is block positive) iff

$$a + b + c \geq p_\theta \text{ and } a \leq 1 \Rightarrow bc \geq (1-a)^2. \tag{4}$$

Proof. (i) $W[a,b,c,\theta]$ is the direct sum of a positive diagonal matrix and the matrix

$$\mathcal{A}_\theta = \begin{pmatrix} a & -e^{i\theta} & -e^{-i\theta} \\ -e^{-i\theta} & a & -e^{i\theta} \\ -e^{i\theta} & -e^{-i\theta} & a \end{pmatrix}$$

which is positive iff $a \geq p_\theta$. The argument for $W[a,b,c,\theta]^\Gamma$ is similar.

(ii) As discussed earlier, a map $\Phi \in L(\mathcal{A}_1, \mathcal{A}_2)$ is positive iff C_Φ is block positive, i.e., iff $\langle C_\phi(x \otimes y), (x \otimes y)\rangle \geq 0$ for all x, y. By appropriate choice of product vectors, the necessity of the conditions in (4) follows. Then a long computation shows C_Φ is block positive if these conditions hold.

□

Lemma 23. *If W and W^Γ are optimal entanglement witnesses, then W is indecomposable.*

Proof. If W is decomposable, we can write $W = P + Q^\Gamma$ with $P, Q \geq 0$. If W is optimal then $P = 0$, and if W^Γ is optimal, then $Q = 0$. Thus if both W and W^Γ are optimal, then W must be indecomposable. □

Ha and Kye characterized spanning properties for the generalized Choi maps, and gave sufficient conditions for them to be indecomposable and exposed.

Theorem 24. *(Thm. 4.1 of Ref. 27) Assume $1 < p_\theta < 2$ and assume $\Phi = \Phi[a,b,c,\theta]$ is positive. Let $W = C_{\Phi[a,b,c,\theta]}$. Then*

(i) W is spanning iff

$$0 \leq a < 1, \quad bc = (1-a^2)$$

(ii) W^Γ is spanning iff either

$$2 - p_\theta \leq a \leq 1, \quad bc = (1-a)^2, \quad a+b+c = p_\theta,$$

or

$$1 \leq a \leq p_\theta, \quad bc = 0, \quad a+b+c = p_\theta.$$

(iii) (Refs. 27 and 30) If

$$2 - p_\theta \leq a < 1, \quad bc = (1-a)^2, \quad a+b+c = p_\theta$$

then Φ is indecomposable and Φ is exposed.

Proof. The fact that (i) or (ii) imply spanning is proven by explicitly finding vectors in Z_W or Z_{W^Γ} respectively that span $H_A \otimes H_B$.

(iii) The given assumptions are equivalent to the combination of (i) and (ii) and thus to $W[a,b,c,\theta]$ and $W[a,b,c,\theta]^\Gamma$ both being spanning. Thus assuming (iii), both W and W^Γ are optimal. By Lemma 23, W is indecomposable. □

Now Ha and Kye[29] can describe the SPA for these generalized Choi maps.

Lemma 25. *Up to a normalizing factor,*

$$\mathrm{SPA}(W[a,b,c,\theta]) = W[p_\theta, p_\theta - a + b, p_\theta - a + c, \theta].$$

Proof. Conveniently, it turns out adding a multiple of $I_m \otimes I_n$ to $W[a,b,c,\theta]$ gives another member of the family (up to a scalar multiple). Let $W_t = (1-t)I_m \otimes I_n + tW$. Then one easily checks that

$$W_t[a,b,c,\theta] = tW[\frac{a_t}{t}, \frac{b_t}{t}, \frac{c_t}{t}, \theta], \tag{5}$$

where $a_t = 1 - t + ta$, $b_t = 1 - t + tb$, $c_t = 1 - t + tc$. Using this and the requirement in Lemma 22 above for positivity of $W[a,b,c,\theta]$ gives the formula for the SPA. □

Kye and Osaka[45] showed that a particular family of these generalized Choi maps have corresponding Choi matrices $W[a,b,c,\theta]$ that are PPT and entangled, as described in the next result.

Theorem 26. *Let $b > 0$, $-\frac{\pi}{3} < \theta < \frac{\pi}{3}$, $\theta \neq 0$. Then $W[p_\theta, b, 1/b, \theta]$ is entangled.*

Proof. Let $W = W[p_\theta, b, 1/b, \theta]$. Kye and Osaka show that there is no product vector $x \otimes y$ in the range of W such that $\bar{x} \otimes y$ is in the range of W^Γ. Thus W fails the range criterion for separability, see Ref. 36. □

Combining the results above, Ha and Kye[29] give a counterexample to the SPA conjecture.

Theorem 27. *Let $W = W[a,b,c,\theta]$.*

(i) If

$$1 < p_\theta < 2, \quad a + b + c \geq p_\theta, \quad 0 \leq a < 1, \quad bc = (1-a)^2 \tag{6}$$

then W is a indecomposable optimal entanglement witness.

(ii) $\lambda_W = \lambda_{W^\Gamma}$ *iff* $\mathrm{SPA}(W)$ *is PPT iff*

$$(p_\theta - a + b)(p_\theta - a + c) = 1. \tag{7}$$

Then $\mathrm{SPA}(W)^\Gamma = \mathrm{SPA}(W^\Gamma)$*, and both* $\mathrm{SPA}(W)$ *and* $\mathrm{SPA}(W^\Gamma)$ *are PPT but not separable.*

(iii) For each choice of θ *with* $\theta \neq \pm\pi/3, \pm\pi$ *there is at least one choice of* a, b, c *such that (i) and (ii) hold, and thus there are examples of an indecomposable optimal entanglement witness whose SPA is PPT but not separable.*

Proof. (i) If (i) holds, then W is block positive by Theorem 22, and since $a < 1 < p_\theta$, then W is not positive. Thus W is an entanglement witness. It is spanning and hence optimal by Theorem 24, and indecomposable by Theorem 23.

(ii) The authors use the conditions for positivity of W and W^Γ from Theorem 22 and the formula (5) to characterize when $\lambda_W = \lambda_{W^\Gamma}$. The remaining statements about $\mathrm{SPA}(W)$ and $\mathrm{SPA}(W^\Gamma)$ follow from Theorem 21. Then the authors apply the results of Kye and Osaka (Theorem 26 above) to show $\mathrm{SPA}(W)$ is not entangled.

(iii) The claim in (iii) follows from a calculation showing that the system of equalities and inequalities give by (6) and (7) and the additional requirement $2 - p_\theta < a$, has one or two solutions for each θ.

□

We remark that if the parameters a, b, c satisfy (6) but not (7), by Theorem 21 W and W^Γ are optimal indecomposable entanglement witnesses such that one or the other of $\mathrm{SPA}(W)$ and $\mathrm{SPA}(W^\Gamma)$ is not PPT, providing additional examples that disprove the SPA conjecture. One such set of parameters is $a = 2 - p_\theta$, $b = c = 1 - a$, $1 < \theta < 2$. If $p_\theta < 4/3$ then $\mathrm{SPA}(W)$ is not separable, if $p_\theta > 4/3$, then $\mathrm{SPA}(W^\Gamma)$ is not separable, and if $p_\theta = 4/3$ then $\mathrm{SPA}(W)$ and $\mathrm{SPA}(W^\Gamma)$ are PPT but not separable.

STØRMER'S DISPROOF OF THE SPA CONJECTURE

Independently, in the same family of optimal entanglement witnesses defined and studied by Ha and Kye, Størmer by different methods proved that there is an entanglement witness that violates the SPA conjecture, and we will sketch Størmer's proof. Størmer's paper in Ref. 72 extends and simplifies some of his arguments from Ref. 71, and we have generally followed the approach in Ref. 72 in our summary here.

Recall that a unit vector $x \in \mathbb{C}^n \otimes \mathbb{C}^n$ is maximally entangled if there are orthonormal bases $b_1, \dots, b_n$ and $c_1, \dots, c_n$ such that $x = \frac{1}{\sqrt{n}} \sum_{i=1}^n b_i \otimes c_i$.

Definition. If $\rho \in M_n \otimes M_n$ is Hermitian, we define

$$S(W) = n \max\{\langle Wx, x\rangle \mid x \in \mathbb{C}^n \otimes \mathbb{C}^n \text{ is a maximally entangled unit vector}\}.$$

(This matches the definition of $S(W)$ in Ref. 72, which is slightly different than that in Ref. 71.)

Note that if W is a density matrix, then $0 \leq S(W) \leq n$, and $S(W) = n$ iff W is a maximally entangled state. Without the scaling factor n, $S(W)$ has been called the maximally entangled fraction of W. Since

$$\|P_x - W\|_2^2 = \mathrm{tr}(P_x - 2WP_x + W^2) = 1 - 2\langle Wx, x\rangle + \mathrm{tr}(W^2)$$

then $\langle Wx, x\rangle$ is maximized for x the maximally entangled state closest to W, so $S(W)$ can be thought of as a measure of the distance from W to the set of maximally entangled states. It is readily verified that $|S(W_1) - S(W_2)| \leq n\|W_1 - W_2\|$ for the operator norm, so S is continuous.

Let $f_1, \dots, f_n$ and $g_1, \dots, g_n$ be orthonormal bases of $\mathbb{C}^n$, and let F_{ij} and G_{kl} be the corresponding systems of matrix units such that $F_{ij} f_p = \delta_{jp} f_i$ and similarly for G_{kl}. The following gives a simple lower bound for $S(W)$ in terms of the matrix for W in the product basis $\{f_i \otimes g_j\}$.

Lemma 28. *Let* $W = \sum_{ijkl} w_{ij,kl} F_{ij} \otimes G_{kl}$, *and* $x = \frac{1}{\sqrt{n}} \sum_i f_i \otimes g_i$. *Then*

$$\langle Wx, x\rangle = \frac{1}{n} \sum_{ij} w_{ij,ij}.$$

Størmer's key tool[71,72] is the following necessary criterion for separability.

Theorem 29. *If* W *is a separable density matrix in* $M_n \otimes M_n$, *then* $S(W) \leq 1$.

Proof. By a straightforward computation making use of Lemma 28, if W_1, W_2 are density matrices, and x is a maximally entangled unit vector, then $\langle (W_1 \otimes W_2)x, x\rangle \leq 1$. Every separable state W is a convex combination of product states, so $\langle Wx, x\rangle \leq 1$. Now $S(W) \leq 1$ follows. □

Theorem 30. *(Ref. 71 and 72) There are values of a, b, c, θ satisfying* (6) *for which $W = W[a, b, c, \theta]$ is an indecomposable optimal entanglement witness with* SPA(W) *not separable.*

Proof. Choose sequences $a_n \to 1$, $\theta_n \to \pi$, $b_n \to 0$, $c_n \to 0$ such that the parameters a_n, b_n, c_n, θ_n satisfy the conditions in (6). (Explicit choices are described in Thm. 10 of Ref. of 71.) Let $\Phi_n = \Phi[a_n, b_n, c_n, \theta_n]$. Then for all n, Φ_n is an optimal positive map. Let

$$W_n = \frac{1}{3(a_n + b_n + c_n)} C_{\Phi_n} = \frac{1}{3(a_n + b_n + c_n)} W[a_n, b_n, c_n, \theta_n].$$

Each W_n is a (normalized) indecomposable optimal entanglement witness.

Note that $\lim \Phi_n = I$, and

$$\lim_n W_n = \frac{1}{3} C_I = \frac{1}{3} P_+.$$

(Recall $\frac{1}{3} P_+ = \frac{1}{3} \sum_{i,j=1}^{3} E_{ij} \otimes E_{ij}$ is the projection onto the maximally entangled vector $\Psi_+ = \frac{1}{\sqrt{3}} \sum_{i=1}^{3} e_i \otimes e_i$, where e_1, e_2, e_3 is the standard basis of $\mathbb{C}^3$.) By the continuity of S and SPA

$$\lim_n S\left(\frac{\text{SPA}(W_n)}{\operatorname{tr}\text{SPA}(W_n)}\right) = S\left(\frac{\text{SPA}(\frac{1}{3}P_+)}{\operatorname{tr}\text{SPA}(\frac{1}{3}P_+)}\right) = S(\frac{1}{3}P_+) = 3 > 1.$$

Thus for n sufficiently large, $\widetilde{W}_n = \frac{\text{SPA}(W_n)}{\operatorname{tr}\text{SPA}(W_n)}$ is a density matrix with $S(\widetilde{W}_n) > 1$. Therefore by Theorem 29, for n sufficiently large, W_n is an indecomposable optimal entanglement witness whose SPA is not separable. □

We remark that in place of $S(a)$, in the arguments above one could instead use $S_0(a) = n\langle a\psi_+, \psi_+\rangle$ where $\psi_+ = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} e_i \otimes e_i$. Then one can explicitly calculate $S_0(\widetilde{W}_n)$ in terms of a_n, b_n, c_n, θ_n and the minimum eigenvalue of $\widetilde{W}_n$ to find n and hence specific parameters a_n, b_n, c_n, θ_n such that $S_0(\widetilde{W}_n) > 1$. Then $S(\widetilde{W}_n) \geq S_0(\widetilde{W}_n) > 1$ so SPA(W_n) is entangled. This is the approach in Ref. 71.

CHRUŚIŃSKI-SARBICKI'S DECOMPOSABLE COUNTEREXAMPLE

After the negative solution of the SPA conjecture with an indecomposable entanglement witness, attention turned to the question of whether an optimal entanglement witness that is decomposable would always have a separable SPA.

By definition, a decomposable entanglement witness has the form $W = P + Q^\Gamma$, with $P, Q \geq 0$. By Lemma 7, if W is optimal then $P = 0$, so $W = Q^\Gamma$. Furthermore, since

SPA(Q^Γ) is a convex combination of Q^Γ and $I \otimes I/d_A d_B$, its partial transpose is positive. Thus the SPA of a decomposable optimal entanglement witness will be PPT (and in dimensions 2×2 or 2×3 will then be separable).

Chruściński and Sarbicki[18] give an example of a decomposable optimal entanglement witness in $M_3 \otimes M_3$ whose SPA is not separable. Their example has the form B^Γ where B is a convex combination of three Bell-like states of the family of nine such states on $M_3 \otimes M_3$ defined in Ref. 4. Let e_1, e_2, e_3 be the standard basis of $\mathbb{C}^3$, and define

$$\Omega_{10} = \frac{1}{\sqrt{3}}(e_1 \otimes e_1 + \omega e_2 \otimes e_2 + \bar{\omega} e_3 \otimes e_3),$$

$$\Omega_{20} = \frac{1}{\sqrt{3}}(e_1 \otimes e_1 + \bar{\omega} e_2 \otimes e_2 + \omega e_3 \otimes e_3),$$

$$\Omega_{11} = \frac{1}{\sqrt{3}}(\bar{\omega} e_1 \otimes e_3 + e_2 \otimes e_1 + \omega e_3 \otimes e_2),$$

where $\omega = e^{2\pi i/3}$ and $\bar{\omega}$ denotes the complex conjugate of ω.

Let P_{10}, P_{20}, P_{11} be the corresponding projections, and for $0 \le \gamma \le 1$ define

$$B_\gamma = \frac{1-\gamma}{2} P_{10} + \frac{1-\gamma}{2} P_{20} + \gamma P_{11}.$$

Then let

$$W_\gamma = 3B_\gamma^\Gamma = \begin{pmatrix} 1-\gamma & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \omega\gamma & \cdot \\ \cdot & \gamma & \cdot & -\frac{1-\gamma}{2} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \bar{\omega}\gamma & \cdot & -\frac{1-\gamma}{2} & \cdot & \cdot \\ \cdot & -\frac{1-\gamma}{2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \bar{\omega}\gamma \\ \cdot & \cdot & \omega\gamma & \cdot & 1-\gamma & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \gamma & \cdot & -\frac{1-\gamma}{2} & \cdot \\ \cdot & \cdot & -\frac{1-\gamma}{2} & \cdot & \cdot & \cdot & \gamma & \cdot & \cdot \\ \bar{\omega}\gamma & \cdot & \cdot & \cdot & \cdot & -\frac{1-\gamma}{2} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \omega\gamma & \cdot & \cdot & \cdot & \cdot & 1-\gamma \end{pmatrix}$$

The authors observe that $W_\gamma = 3B_\gamma^\Gamma$ isn't positive by observing it has a 3×3 direct summand with a negative eigenvalue. Thus W_γ is an entanglement witness. Then the authors give a direct proof that W_γ has the spanning property for all $0 < \gamma < 1$, hence is optimal.

Theorem 31. *(Ref. 18) For γ in an interval containing $3/4$, W_γ is a decomposable optimal entanglement witness whose structural physical approximation is entangled.*

Proof. To find values of γ for which $\mathrm{SPA}(W_\gamma)$ is not separable, the authors make use of the realignment criterion. For a matrix ρ, Chen and Wu[6] defined a "realigned" matrix $R(\rho)$, and showed that if ρ is separable, then $\|\operatorname{tr} R(\rho)\|_1 = (\operatorname{tr}(R(\rho)R(\rho)^\dagger))^{1/2} \leq \operatorname{tr} R(\rho)$. (As they remark, their test is equivalent to Rudolph's[59] cross norm separability criterion.)

If $-\lambda_\gamma$ is the minimal eigenvalue of W_γ, let $Q_\gamma = W_\gamma + \lambda_\gamma I \otimes I = \mathrm{SPA}_0(W_\gamma)$. Here $R(Q_\gamma)R(Q_\gamma)^\dagger$ is a direct sum of three 3 x 3 submatrices, and Chruściński and Sarbicki find an explicit expression for $\operatorname{tr} R(Q_\gamma)R(Q_\gamma)^\dagger$ in terms of γ and λ_γ. They use this to show that for $\gamma = 3/4$, Q_γ fails the realignment criterion, and thus is not separable. Thus $\mathrm{SPA}_0(W_\gamma)$ and $\mathrm{SPA}(W_\gamma)$ are not separable. (They show numerically that the same conclusion holds for an range of values of γ around $3/4$.) Thus the SPA conjecture also fails when restricted to decomposable entanglement witnesses.

$\square$

In conclusion, after many examples were found supporting the SPA conjecture, indecomposable and decomposable families of counterexamples now have been found.

Acknowledgement The author thanks Mary Beth Ruskai for a variety of suggestions that substantially improved this review.

REFERENCES

[1] R. Augusiak, J. Bae, Ł. Czekaj, and M. Lewenstein. On structural physical approximations and entanglement breaking maps. *Journal of Physics A-Mathematical and Theoretical*, 44(18), 2011.

[2] R. Augusiak, J. Tura, and M. Lewenstein. A note on the optimality of decomposable entanglement witnesses and completely entangled subspaces. *Journal of Physics A-Mathematical and Theoretical*, 44(21), 2011.

[3] G. P. Barker and D. Carlson. Cones of diagonally dominant matrices. *Pacific Journal of Mathematics*, 57(1):15–32, 1975.

[4] B. Baumgartner, B. C. Hiesmayr, and H. Narnhofer. The geometry of bipartite qutrits including bound entanglement. *Physics Letters A*, 372(13):2190–2195, 2008.

[5]H. P. Breuer. Optimal entanglement criterion for mixed quantum states. *Physical Review Letters*, 97(8), 2006.

[6]K. Chen and L.-A. Wu. A matrix realignment method for recognizing entanglement. *Quantum Information & Computation*, 3(3):193–202, 2003.

[7]S. J. Cho, S.-H. Kye, and S. G. Lee. Generalized choi maps in 3-dimensional matrix algebra. *Linear Algebra and Its Applications*, 171:213–224, 1992.

[8]M. D. Choi. Completely positive linear maps on complex matrices. *Linear Algebra and Its Applications*, 10(3):285–290, 1975.

[9]M. D. Choi. Positive linear maps, in *Proc. Sympos. Pure Math.*, Vol. 38, Part 2, 1982, pp. 583–590].

[10]M. D. Choi. Positive semidefinite biquadratic forms. *Linear Algebra and Its Applications*, 12(2):95–100, 1975.

[11]M.-D. Choi. Some assorted inequalities for positive linear maps on c*-algebras. *J. Operator Theory*, 4:271–285, 1980.

[12]M. D. Choi and T. Y. Lam. Extremal positive semidefinite forms. *Mathematische Annalen*, 231(1):1–18, 1977.

[13]D. Chruściński and J. Pytel. Constructing optimal entanglement witnesses. II. witnessing entanglement in 4n x 4n systems. *Physical Review A*, 82(5), 2010.

[14]D. Chruściński and J. Pytel. Optimal entanglement witnesses from generalized reduction and robertson maps. *Journal of Physics A-Mathematical and Theoretical*, 44(16), 2011.

[15]D. Chruściński, J. Pytel, and G. Sarbicki. Constructing optimal entanglement witnesses. *Physical Review A*, 80(6), 2009.

[16]D. Chruściński and G. Sarbicki. Exposed positive maps: a sufficient condition. *Journal of Physics A-Mathematical and Theoretical*, 45(11), 2012.

[17]D. Chruściński and G. Sarbicki. Exposed positive maps in $M_4(C)$. *Open Systems & Information Dynamics*, 19(3), 2012.

[18]D. Chruściński and G. Sarbicki. Disproving the conjecture on the structural physical approximation to optimal decomposable entanglement witnesses. *Journal of Physics A-Mathematical and Theoretical*, 47(19), 2014.

[19]D. Chruściński and G. Sarbicki. Entanglement witnesses: construction, analysis and classification. *Journal of Physics A-Mathematical and Theoretical*, 47(48), 2014.

[20]D. Chruściński and F. A. Wudarski. Geometry of entanglement witnesses for two qutrits.

Open Systems & Information Dynamics, 18(4):375–387, 2011.

[21] J. de Pillis. Linear transformation which preserve hermitian and positive semidefinite operators. *Pacific Journal of Mathematics*, 23(1):129–137, 1967.

[22] E. G. Effros. Order ideals in a C*-algebra and its dual. *Duke Math. J.*, 30:391–412, 1963.

[23] M. H. Eom and S.-H. Kye. Duality for positive linear maps in matrix algebras. *Mathematica Scandinavica*, 86(1):130–142, 2000.

[24] J. Fiurásek. Structural physical approximations of unphysical maps and generalized quantum measurements. *Physical Review A*, 66(5), 2002.

[25] K. C. Ha and S.-H. Kye. Entanglement witnesses arising from exposed positive linear maps. *Open Systems & Information Dynamics*, 18(4):323–337, 2011.

[26] K. C. Ha and S.-H. Kye. One-parameter family of indecomposable optimal entanglement witnesses arising from generalized choi maps. *Physical Review A*, 84(2), 2011.

[27] K. C. Ha and S.-H. Kye. Entanglement witnesses arising from Choi type positive linear maps. *Journal of Physics A-Mathematical and Theoretical*, 45(41), 2012.

[28] K. C. Ha and S.-H. Kye. Optimality for indecomposable entanglement witnesses. *Physical Review A*, 86(3), 2012.

[29] K. C. Ha and S.-H. Kye. The structural physical approximations and optimal entanglement witnesses. *Journal of Mathematical Physics*, 53(10), 2012.

[30] K. C. Ha and S.-H. Kye. Exposedness of choi-type entanglement witnesses and applications to lengths of separable states. *Open Systems & Information Dynamics*, 20(4), 2013.

[31] K. C. Ha and H. Yu. Optimal indecomposable witnesses without extremality or the spanning property. *Journal of Physics A-Mathematical and Theoretical*, 45(39), 2012.

[32] W. Hall. A new criterion for indecomposability of positive maps a new criterion for indecomposability of positive maps a new criterion for indecomposability of positive maps. *J. Phys. A*, 39(45):14119–14131, 2006.

[33] M. Horodecki and P. Horodecki. Reduction criterion of separability and limits for a class of distillation protocols. *Phys. Rev. A*, 59:4206–4216, 1999.

[34] M. Horodecki, P. Horodecki, and R. Horodecki. Separability of mixed states: Necessary and sufficient conditions. *Physics Letters A*, 223(1-2):1–8, 1996.

[35] M. Horodecki, P. Shor, and M. Ruskai. Entanglement breaking channels. *Reviews in Mathematical Physics*, 15(6):629–641, 2003.

[36] P. Horodecki. Separability criterion and inseparable mixed states with positive partial

transposition. *Physics Letters A*, 232:333–339, 1997.

[37]P. Horodecki. From limits of quantum operations to multicopy entanglement witnesses and state-spectrum estimation. *Physical Review A*, 68(5), 2003.

[38]P. Horodecki and A. Ekert. Method for direct detection of quantum entanglement. *Physical Review Letters*, 89(12), 2002.

[39]P. Horodecki, J. Smolin, B. Terhal, and A. Thapliyal. Rank two bipartite bound entangled states do not exist. *Theoretical Computer Science*, 292:589–596, 2003.

[40]A. Jamiołkowski. Linear transformations which preserve trace and positive semidefiniteness of operators. *Reports on Mathematical Physics*, 3(4):275–278, 1972.

[41]J. K. Korbicz, M. L. Almeida, J. Bae, M. Lewenstein, and A. Acin. Structural approximations to positive maps and entanglement-breaking channels. *Physical Review A*, 78(6), 2008.

[42]K. Kraus. General state changes in quantum theory. *Ann. Phys.*, 64:311–335, 1971.

[43]K. Kraus. Operations and Effects in the Hilbert Space Formulation of Quantum Theory, in *Lecture Notes in Physics* Vol. 29, 1974, pp. 206–229.

[44]S.-H. Kye. Facial structures for various notions of positivity and applications to the theory of entanglement. *Reviews in Mathematical Physics*, 25(2), 2013.

[45]S.-H. Kye and H. Osaka. Classification of bi-qutrit positive partial transpose entangled edge states by their ranks. *Journal of Mathematical Physics*, 53(5), 2012.

[46]S.-H. Kye. Facial structures for positive linear maps between matrix algebras. *Can. Math. Bull. 39, 74 (1996).*, 39:74–, 1996.

[47]M. Lewenstein, B. Kraus, J. I. Cirac, and P. Horodecki. Optimization of entanglement witnesses. *Physical Review A*, 62(5), 2000.

[48]W. A. Majewski and T. Tylec, Comment on "Channel-state duality". *Physical Review A*, 88:026301, 2013.

[49]M. Marciniak. Rank properties of exposed positive maps. *arxiv:1103.3497v1*, 2011.

[50]S. Fu, M. Jiang, S. Luo. Channel-state duality. *Phys. Rev. A.*, 87:022310, 2013.

[51]H. Osaka. Indecomposable positive maps in low dimensional matrix algebras. *Linear Algebra Appl*, 153:73, 1991.

[52]H. Osaka. Absolutely indecomposable positive operators. *Linear Algebra and Its Applications*, 186:45, 1993.

[53]V. Paulsen and F. Shultz. Complete positivity of the map from a basis to its dual basis.

J. Math. Phys., 54, 2013.

[54]A. Peres. Separability Criterion for Density Matrices. *Phys. Rev. Lett.*, 77:1413–1415, 1966.

[55]R. T. Prosser. On the ideal structure of operator algebras. *Memoirs of the American Mathematical Society*, 45, 1963.

[56]X. F. Qi and J. C. Hou. Positive finite rank elementary operators and characterizing entanglement of states. *Journal of Physics A: Mathematical and Theoretical*, 44(21):215305, 2011.

[57]X. F. Qi and J. C. Hou. Characterization of optimal entanglement witnesses. *Physical Review A*, 85(2), 2012.

[58]R. T. Rockafellar. *Convex Analysis*. Princeton University Press, 1970.

[59]O. Rudolph. Further results on the cross norm. *arxiv:quant-ph/0202121v1*, 2002.

[60]G. Sarbicki. General theory of detection and optimality. *arxiv:0995.0778v2*, 2011.

[61]G. Sarbicki and D. Chruściński. A class of exposed indecomposable positive maps. *Journal of Physics A-Mathematical and Theoretical*, 46(1), 2013.

[62]R. Schneider. *Convex bodies: The Brunn-Minkowski Theory*. Cambridge University Press, 2013.

[63]Ł. Skowronek and E. Størmer. Choi matrices, norms and entanglement associated with positive maps on matrix algebras. *Journal of Functional Analysis*, 262(2):639–647, 2012.

[64]Ł. Skowronek, E. Størmer, and K. Zyczkowski. Cones of positive maps and their duality relations. *Journal of Mathematical Physics*, 50(6), 2009.

[65]E. Størmer. Positive linear maps of operator algebras. *Acta Math.*, 110:233–278, 1963.

[66]E. Størmer. Decomposition of positive projections on C*-algebras. *Mathematische Annalen*, 247(1):21–41, 1980.

[67]E. Størmer. Decomposable positive maps on C*-algebras. *Proceedings of the American Mathematical Society*, 86(3):402–404, 1982.

[68]E. Størmer. Separable states and positive maps. *Journal of Functional Analysis*, 254(8):2303–2312, 2008.

[69]E. Størmer. Mapping cones of positive maps. *Mathematica Scandinavica*, 108(2):223–232, 2011.

[70]E. Størmer. *Positive linear maps of operator algebras*. Springer-Verlag, 2013.

[71]E. Størmer. Separable states and the structural physical approximation of a positive map.

Journal of Functional Analysis, 264(9):2197–2205, 2013.

[72]E. Størmer. Maximally entangled states, and positive maps. *preprint*, 2015.

[73]K. Tanahashi and J. Tomiyama. Indecomposable maps in matrix algebras. *Can. Math. Bull.*, 31:308, 1988.

[74]B. M. Terhal. Bell inequalities and the separability criterion. *Physics Letters A*, 271(5-6):319–326, 2000.

[75]B. H. Wang and D. Y. Long. Structural physical approximations and entanglement witnesses. *Physical Review A*, 87(6), 2013.

[76]R. Werner. Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden variable model. *Phys. Rev. A*, 40:4277–4281, 1989.

[77]S. L. Woronowicz. Positive maps of low dimensional matrix algebras. *Rep. Math. Phys*, 10:165–183, 1976.

[78]J. P. Zwolak and D. Chruściński. Recurrent construction of optimal entanglement witnesses for 2n-qubit systems. *Physical Review A*, 89(5), 2014.